Assignment 2 Solution

1. $f(z) = u + iv \qquad z = x + iy = r(\cos\theta + i\sin\theta)$

$$f'(z) = u_x + i v_x = v_y - i u_y$$

$$u_r = \frac{du}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial u}{\partial r}\cdot\frac{\partial y}{\partial r} = u_x \cdot \cos\theta + u_y \sin\theta$$

$$u_\theta = u_x(-r\sin\theta) + u_y \cdot r\cos\theta$$

$$\therefore\ u_x = u_r \cdot \cos\theta - \frac{1}{r} u_\theta \sin\theta$$

$$u_y = u_r \cdot \sin\theta + \frac{1}{r} u_\theta \cos\theta$$

$$v_r = v_x \cos\theta + v_y \sin\theta = -u_y\cos\theta + u_x \sin\theta = \frac{u_\theta}{-r}$$

$$v_\theta = v_x(-r\sin\theta) + v_y r\cos\theta = u_y \cdot r\sin\theta + u_x \cdot r\cos\theta = r u_r$$

$$f'(z_0) = u_x + i v_x = u_x - i u_y = \left(u_r\cos\theta_0 - \frac{1}{r_0} u_\theta \sin\theta_0\right) - i\left(u_r \sin\theta_0 + \frac{1}{r_0} u_\theta \cos\theta_0\right)$$

$$= u_r(\cos\theta_0 - i\sin\theta_0) - \frac{1}{r_0} u_\theta(\sin\theta_0 + i\cos\theta_0) = u_r(\cos\theta_0 - i\sin\theta_0) + v_r(\sin\theta_0 + i\cos\theta_0)$$

$$= u_r \cdot e^{-i\theta_0} + i v_r e^{-i\theta_0} = e^{-i\theta_0}\,\underline{\underline{(u_r + i v_r)}}$$

$$= e^{-i\theta_0}\left(\frac{v_\theta}{r} - i\frac{u_\theta}{r}\right)$$

$$= \frac{1}{r e^{i\theta_0}}(v_\theta - i u_\theta)$$

$$= \underline{\underline{\frac{-i}{z_0}(u_\theta + i v_\theta)}}$$

2 (a) As $z \neq 0$

$$f(z) = f(x+iy) = (1+i)\frac{\operatorname{Im}(x+iy)^2}{|x+iy|^2} = (1+i)\frac{2xy}{x^2+y^2}$$

$$\therefore u(x,y) = \frac{2xy}{x^2+y^2} \qquad v(x,y) = \frac{2xy}{x^2+y^2}$$

$$\frac{\partial v}{\partial x}(0,0) = \frac{\partial u}{\partial x}(0,0) = \lim_{\Delta x \to 0} \frac{\frac{2\cdot\Delta x\cdot 0}{\Delta x^2+0^2} - 0}{\Delta x} = 0$$

$$\frac{\partial v}{\partial y}(0,0) = \frac{\partial u}{\partial y}(0,0) = \lim_{\Delta y \to 0} \frac{\frac{2\cdot 0\cdot\Delta y}{0^2+\Delta y^2} - 0}{\Delta y} = 0$$

$$\therefore \frac{\partial u}{\partial x}(0,0) = \frac{\partial v}{\partial y}(0,0) \qquad \frac{\partial u}{\partial y}(0,0) = -\frac{\partial v}{\partial x}(0,0)$$

$\therefore$ C-R equations are satisfied at $z=0$

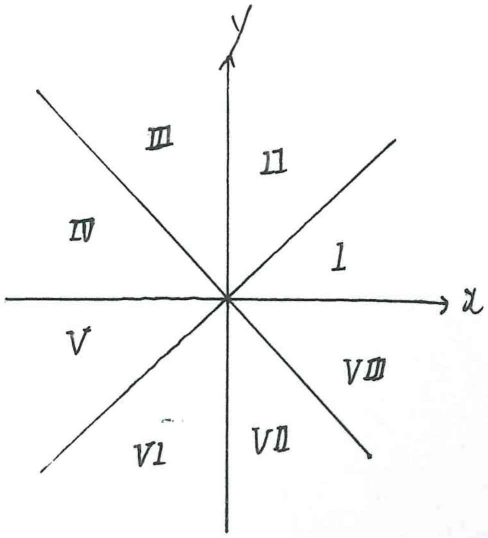
(b) Let $\Delta z = \Delta x$

$$\lim_{\Delta z \to 0} \frac{f(0+\Delta z) - f(0)}{\Delta z} = \lim_{\Delta x \to 0} \frac{0-0}{\Delta z} = 0$$

let $\Delta x = \Delta y$ $\quad \Delta x \to 0$

$$\lim_{\Delta z \to 0} \frac{f(0+\Delta x+\Delta y i) - f(0)}{\Delta z} = \lim_{\Delta x \to 0} \frac{(1+i)\frac{2\Delta x\cdot\Delta x}{\Delta x^2+\Delta x^2} - 0}{(1+i)\Delta x} = \lim_{\Delta x \to 0} \frac{1}{\Delta x} = \infty$$

So $f(z)$ is not differentiable at 0.



$f(z) = |x^2-y^2| + 2i|xy|$

$\mathrm{I} \cup \mathrm{V} := \{(x,y) \mid xy>0,\ x^2>y^2\}$

$f(z) = x^2-y^2+2ixy$ $\quad u_x = 2x \quad u_y = -2y \quad v_x = 2y \quad v_y = 2x$

$\therefore$ $f(z)$ is analytic

$\mathrm{IV} \cup \mathrm{VIII} := \{(x,y) \mid xy<0,\ x^2>y^2\}$

$f(z) = x^2-y^2-2ixy$ $\quad u_x = 2x \quad u_y = -2y \quad v_x = -2y \quad v_y = -2x$

$\therefore$ $f(z)$ is NOT analytic

$\mathrm{II} \cup \mathrm{VI} := \{(x,y) \mid xy>0,\ x^2<y^2\}$

$f(z) = y^2-x^2+2ixy$ $\quad u_x = -2x, \quad u_y = 2y \quad v_x = 2y \quad v_y = 2x$

$\therefore$ $f(z)$ is NOT analytic

$\mathrm{III} \cup \mathrm{VII} := \{(x,y) \mid xy<0,\ x^2<y^2\}$

$f(z) = y^2-x^2-2ixy$ $\quad u_x = -2x \quad u_y = 2y \quad v_x = -2y \quad v_y = -2x$

$\therefore$ $f(z)$ is analytic

① $x_0 = y_0 = 0$

$$\lim_{z\to z_0}\frac{f(z)-f(z_0)}{z-z_0} = \lim_{\Delta z\to 0}\frac{|\Delta x^2-\Delta y^2|+2i|\Delta x\Delta y|}{\Delta x+i\Delta y}$$

$$= \lim_{\Delta z\to 0}\frac{(|\Delta x^2-\Delta y^2|+2i|\Delta x\Delta y|)}{\Delta x^2+\Delta y^2}(\Delta x-i\Delta y)$$

meanwhile $\frac{|\Delta x^2-\Delta y^2|}{\Delta x^2+\Delta y^2}\le 1$ $\quad \left|\frac{2i|\Delta x\Delta y|}{\Delta x^2+\Delta y^2}\right| = \frac{2|\Delta x\Delta y|}{\Delta x^2+\Delta y^2}\le 1$

$\therefore \lim_{z\to z_0}\frac{f(z)-f(z_0)}{z-z_0} = 0$ $\quad \therefore$ $\boxed{f(z) \text{ is differentiable at } (0,0)}$

② $y_0 = 0 \quad x_0 > 0$

$$\lim_{z\to z_0}\frac{f(z)-f(z_0)}{z-z_0} = \lim_{\Delta z\to 0}\frac{|(x_0+\Delta x)^2-\Delta y^2|+2i|(x_0+\Delta x)\cdot\Delta y|-x_0^2}{\Delta x+i\Delta y} \leftarrow *$$

let $\Delta y = 0$ $\quad * = \lim_{\Delta x\to 0}\frac{|(x_0+\Delta x)^2|-x_0^2}{\Delta x} = \lim_{\Delta x\to 0}\frac{x_0^2+2x_0\Delta x+\Delta x^2-x_0^2}{\Delta x} = 2x_0$

let $\Delta x = 0$ $\quad * = \lim_{\Delta y\to 0^-}\frac{|x_0^2-\Delta y^2|+2i|x_0\Delta y|-x_0^2}{i\Delta y} = \frac{x_0^2-\Delta y^2-2ix_0\Delta y-x_0^2}{i\Delta y} = -2x_0$

$2x_0 \neq -2x_0$

$\therefore$ $f(z)$ is NOT differentiable at $(x_0, 0)$

③ $y_0 = 0 \quad x_0 < 0$ $\qquad$ Use the same method in ②, we know

$y_0 > 0 \quad x_0 = 0$

$y_0 < 0 \quad x_0 = 0$

$\boxed{f(z) \text{ is NOT differentiable at } (x_0, y_0) \text{ As } x_0 y_0 = 0 \quad x_0^2+y_0^2 \neq 0}$

④ $x_0^2 = y_0^2 \quad x_0, y_0 > 0$

$$\lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0} = \lim_{\Delta z \to 0} \frac{|(x_0+\Delta x)^2 - (y_0+\Delta y^2))| + 2i|(x_0+\Delta x)(y_0+\Delta y))| - 2i|x_0 y_0|}{\Delta x + \Delta y i} \quad *$$

let $\Delta x = \Delta y$ $\quad * = \lim_{\Delta x \to 0} \frac{2i|(x_0+\Delta x)^2| - 2i|x_0^2|}{\Delta x + \Delta x i} = \lim_{\Delta x\to 0} \frac{2i(x_0^2 + 2x_0\Delta x + \Delta x^2 - x_0^2)}{\Delta x(1+i)}$

$$= \lim_{\Delta x \to 0} \frac{2i}{(1+i)}(2x_0 + \Delta x) = \frac{2i}{(1+i)} \cdot 2x_0 = \frac{4ix_0}{1+i}$$

let $\Delta y = -\Delta x$, $\Delta x \to 0^-$ $\quad * = \lim_{\Delta x\to 0^-} \frac{|(x_0+\Delta x)^2 - (x_0-\Delta x)^2| + 2i|(x_0+\Delta x)(x_0-\Delta x)| - 2i x_0^2}{\Delta x - \Delta x i}$

$$= \lim_{\Delta x \to 0^-} \frac{|4x_0\Delta x| + 2i(x_0^2 - \Delta x^2) - 2ix_0^2}{\Delta x - \Delta x i}$$

$$= \lim_{\Delta x\to 0^-} \frac{|4x_0\Delta x| - 2i\Delta x^2}{\Delta x(1-i)} = \lim_{\Delta x \to 0^-} \frac{-4x_0\Delta x}{\Delta x(1-i)} = \frac{-4x_0}{1-i}$$

$$\frac{4ix_0}{1+i} \neq \frac{-4x_0}{1-i}$$

Use the same argument, we know

$\boxed{f(z) \text{ is not differentiable at } (x_0, y_0), \text{ as } x_0^2 = y_0^2 \quad x_0 \neq 0}$

4 (a) $z(t) = 2(1-t) + 2it \ , \quad 0 \le t \le 1$

$z'(t) = -2 + 2i$

$$\int_\gamma z^2 dz = \int_0^1 [2(1-t)+2it]^2(-2+2i)dt = 8\int_0^1 [(1-t)+it]^2(-1+i)\,dt$$

$$= 8(-1+i)\frac{1}{i-1}\frac{[(1-t)+it]^3}{3}\Big|_0^1 = 8\left(\frac{-i}{3}-\frac{1}{3}\right) = -\frac{(8+8i)}{3}$$

(b) $z(\theta) = 2e^{i\theta} \ , \quad 0 \le \theta \le \frac{\pi}{2}$

$z'(\theta) = 2ie^{i\theta}$

$$\int_\gamma z^2 dr = \int_0^{\frac{\pi}{2}} 4e^{2i\theta}\cdot 2ie^{i\theta}d\theta = 8i\int_0^{\frac{\pi}{2}} e^{3i\theta}d\theta = \frac{8i}{3i}\ e^{3i\theta}\Big|_0^{\frac{\pi}{2}}$$

$$= -\frac{8+8i}{3}$$

5 $L = \frac{2\pi\cdot 2}{4} = \pi$

$|z^2-1| \geq |z^2|-1 = 3 \quad \therefore \frac{1}{|z^2-1|} \le \frac{1}{3}$

$$\therefore \left|\int_C \frac{dz}{z^2-1}\right| \le \frac{1}{3}\cdot\pi$$

6 $L = \frac{2\pi\cdot R}{2} = \pi R$

$|z|^2 = R^2 \qquad |z^6+1| \geq |z^6|-1 = R^6-1$

$$\therefore \left|\int_{C_R}\frac{z^2}{z^6+1}dz\right| \le \pi R\cdot\frac{R^2}{R^6-1} = \frac{\pi R^3}{R^6-1} \qquad \lim_{R\to+\infty}\frac{\pi R^3}{R^6-1} = \lim_{R\to+\infty}\frac{\pi}{R^3-\frac{1}{R^3}} = 0$$

$$\therefore \lim_{R\to+\infty}\int_{C_R}\frac{z^2}{z^6+1}dz = 0$$